Mixed notation and mathematical writing in Danish upper secondary school

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The paper points to the emergence of the phenomenon "mixed notation" as a result of the use of Computer Algebra Systems (CAS) in upper secondary education. We provide an "existence proof" of mixed notation consisting in a student's written answer to an exam question from the Danish national mathematics exam as an instance of mathematical writing. Based on a qualitative analysis of the student's written answer, we discuss how mixed notation emerges as a result of the technological context and the classroom culture. Finally, we argue that mixed notation calls for awareness in regard to how the ongoing transformation of written mathematical activities, as a result of using CAS, influence students' mathematical learning and identity work.

Keywords: Mathematical writing, CAS, mathematical notation.

Introduction

The increased use of digital media for students' mathematical writing does influence their mathematical work. One example is that students now hand in various printouts and computer files rather than handwritten assignments. This is also the case at the national exams in Danish upper secondary education. These changes in the materiality of mathematical writing contribute to a change in the specific notations and diagrams used by the students to signify mathematical objects and processes. However, such a change might not be as superfluous and innocent as one might think. Several theoretical and empirical contributions suggest that there is a complex interplay between notation and other representations and the cognitive processes, both in general and in relation to upper secondary mathematics (e.g. Kieran & Drijvers, 2006). One reason is that the use of different media for writing in the mathematics might amplify and/or reduce the use of specific semiotic resources in students' responses, and that such semiotic resources are associated with different cognitive processes (Duval, 2006; Mariotti, 2002).

In Danish upper secondary education there are a growing proportion of students who use Computer Algebra Systems (CAS) as a medium for writing in mathematics. Since CAS have a slightly different mathematical notation, and strong interactive abilities (Lagrange, 2005), including the capability to black-box certain mathematical processes (Nabb, 2010), it is worth focusing on how CAS-related notation affects mathematics learning. Currently it seems likely that CAS eventually take over as the common mathematical medium in Danish upper secondary education. Hence, knowledge about its influence on notation and learning is important – and this even more so, since we know from research on literacy that a change in medium and language "transmitting" knowledge will affect other dimensions of learning, education, and competence. As an example, Kolstø (2010) and Vollmer (2009) point out the importance of learning a subject's subject-specific-language. In their view learning the subject-specific-language is closely related to learning the subjects' certain ways of thinking and doing. This point is also made in relation to mathematical discourse and learning (Darragh, 2016; O'Halloran, 2005 Sfard, 2008; Sfard & Prusak, 2005; Steentoft & Valero, 2009).

Our ambition with the present paper is to show how classical algebraic notation and CAS-related notation is entangled by students in upper secondary education. We present this "mixed notation" as a phenomenon of relevance to us as mathematics educators and present an illustrative case of one student's mathematical writing in order to aim at a first characterization of the phenomenon. Hence, this paper is not to be viewed as a traditional empirical research study, but rather as theoretical piece providing an "existence proof" and characterization of an observed phenomenon.

Theoretical framework

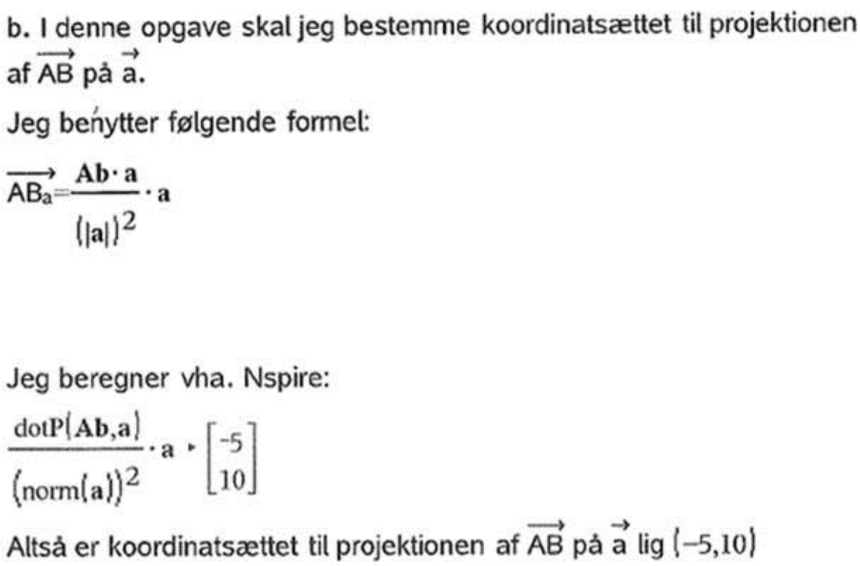
We apply the instrumental approach to the use of technology in mathematics education and hence we see technology as mediating artifacts in mathematical activities (Artigue, 2002; Trouche, 2005). As an outset for looking at mixed notation we will use the distinction between *epistemic* and *pragmatic mediations* (Artigue, 2002; Lagrange 2005; Trouche, 2005), and augment it with a consideration of students' identity work. An epistemic mediation is directed towards the user's cognitive system; the tool is used to create a different understanding or to support learning. For example, Lagrange (2005) refers to experimental uses of computers, e.g. in relation to students' mathematical concept formation. In contrast, a pragmatic mediation is directed towards something external to the user; the tool is used to create a difference in the external world. Lagrange (2005) refers to the mathematical technique of "pushing buttons". We augment this understanding of the roles of technology with the concept of identity – or what we will argue that we meaningfully can refer to as *identity directed mediation*.

Our focus on identity is based on socio-cultural perspectives of teaching-learning processes. Ivanič (2006) argues that students' learning is closely linked to processes of identification, meaning the extent to which students identify with the values, beliefs, goals, and activities that prototypical participants in the learning activities represent. The view that identification is an important factor in learning is shared by a number of scholars (e.g. Gee, 2001). In the words of Hyland (2009, p. 70), "identity is something we do; not something we have." All of us do identity all the time, and this doing has been coined as *identity work* by Gee (2003). In this way, identity can be understood as negotiated ways of participating in different social groups, cultures and institutions, and of course identity work is *mediated* by the tools, technologies and representational systems at hand. Hence, we apply a theoretical lens based in the same sociocultural outset as the instrumental approach, but we include identity work as a third type of mediation (in addition to those of epistemic and pragmatic).

The basic insight from the instrumental approach is that there is a dialectics between artefact and individual, when adopting artefacts as tools for work. In relation to identity this means that students, on the one hand, are expected to use artefacts (digital tools, forms of notation, etc.) to perform identity work in ways not foreseen by teachers and technology developers. And, on the other hand, that these artefacts change and affect the students' identity work.

Presenting an illustrative case of mixed notation

The following illustrative case is taken from a longitudinal study of students' mathematical writing in the subject of mathematics. This field study took place over a two-year period (2011-2013) and consisted of several studies from different types of Danish upper secondary education. One of the findings of the study was that CAS are increasingly used as a medium for students' mathematical writing (Iversen, 2014). We present an excerpt taken from the student Anna's reply to a written examination in mathematics (see Figure 1). Anna's answer serves as an example of key differences between classical algebraic notation and CAS notation as well as an example of their entanglement. Notice how the central formula used in the solution of the task is written up in two different ways (line 4 and line 6 in the excerpt of Figure 1).


b. I denne opgave skal jeg bestemme koordinatsættet til projektionen
af $\overrightarrow{AB}$ på $\vec{a}$.
Jeg benytter følgende formel:
$\overrightarrow{AB_a} = \frac{\mathbf{Ab} \cdot \mathbf{a}}{(|\mathbf{a}|)^2} \cdot \mathbf{a}$

Jeg beregner vha. Nspire:
$\frac{\mathrm{dotP}(\mathbf{Ab},\mathbf{a})}{(\mathrm{norm}(\mathbf{a}))^2} \cdot \mathbf{a} \blacktriangleright \begin{bmatrix} -5 \\ 10 \end{bmatrix}$
Altså er koordinatsættet til projektionen af $\overrightarrow{AB}$ på $\vec{a}$ lig (−5,10)

Figure 1: Excerpt of Anna's written answer to a sub-question (7.b) in a written exam. The text reads: (line 1-2) "b. In this task I am to decide the coordinates to the projection of $\overrightarrow{AB}$ on $\vec{a}$." (line 3) "I use the following formula:" (line 5) "I calculate using Nspire:" (line 7) "Hence, the coordinates to the projection of $\overrightarrow{AB}$ on $\vec{a}$ equals (-5,10)**"**

In the first two lines Anna paraphrases the formulation of the task. In line 3 she indicates the mathematical formula that she is going to use to solve the task. The formula is firstly written with algebraic notation as it typically occurs in textbooks and lecture notes as well as in formula tables and task formulations. A deviation from this is that Anna indicates the vectors included in the formula (to the right of the equal sign) in bold (**Ab** and **a**). As seen in the first two lines, i.e. the paraphrasing of the exam text by the Danish Ministry of Education, vectors are conventionally written using a notation of small horizontal arrows above letters, $\overrightarrow{AB}$ and $\vec{a}$.

We cannot know why Anna uses the notation of putting vectors in bold (typically not taught in Danish upper secondary school), but it is probably because she previously defined $\vec{a}$ in her CAS TI-Nspire – and TI-Nnspire uses the notation **a** for a vector. (Notice that the problem Anna is working on is a sub-problem of a larger set of problems, where the vector $\vec{a}$ has been used previously). In that sense, one can argue that Anna is using elements of CAS notation already in the formula in line 4 of Figure 1. We do, however, consider this formula mainly as an example of algebraic notation. This makes sense if we compare the formula with the version of the same formula shown on line 6. In fact, this illustrates some key differences between algebraic notation and CAS notation. In the latter case, **Ab · a** is replaced by dotP(**Ab**, **a**), and norm(**a**) is used instead of the typical $|\vec{a}|$. In addition, Anna is using a small triangle between the formula and the calculated result, whereas in algebraic notation one would typically use an equal sign (=). The transition from line 4, where one single notational element from CAS moves into the students' writing, to more full blown CAS notation in line 6 captures what we mean by *mixed notation*. Namely, the fact that elements from CAS notational conventions are imported into students written products.

How to understand the phenomenon of mixed notation

We now analyze the case of Anna presented above and aim at characterizing some relevant dimensions in the phenomenon of *mixed notation*. In the case of Anna the use of CAS notation is not just a meaningless markup language used to document her work with a CAS tool, which then ideally should be translated back to classical algebraic notation. Rather we see elements of CAS notation being used as a natural part of the communication around Anna's solution of the problem. It appears that the two types of notation assist each other in the construction of Anna's argument – and hence also in her *identity work* (Gee, 2003) as someone doing mathematics (see also Iversen, 2013). When Anna for example indicates the length of the vector $\vec{a}$, not by the algebraic notation $|\vec{a}|$, but by the more CAS-oriented and keyboard friendly norm(**a**), she is in part reporting on her CAS-based calculations. But at the same time she is also transforming her communication with the teacher to include CAS notation. These types of notational transformations are performed by students and are to a large extent accepted – sometimes even endorsed – by mathematics teachers in Danish upper secondary education (Iversen, 2014).

Furthermore, elements of CAS notation and CAS use are contributing to the shaping and molding of the mathematical identities of the students (Iversen, Misfeldt, & Jankvist, in progress). That mixed notation is part of students' identity work and students' learning – we argue – goes counter to a first approximation of the role of CAS notation in upper secondary school students' work, namely as a technical discourse related only to the instrument. This first approximation of mixed notation, as a superfluous byproduct of the technical means that students bring into play, would suggest that skilled students take out the CAS aspects in the theoretical parts of their communication with their teacher and only provide the teacher with a genuine algebraic translation of the CAS work. And this is not the case (Iversen, Misfeldt & Jankvist, in progress).

In fact, it is obvious that one of the affordances of the mixed notation is that the students are able to report on their CAS-based work in a direct manner. Line 6 in the example (Figure 1) does have aspects of that in it. However, we do not see students or teachers unanimously suggesting that CAS-related mathematical notation should only be used for reporting the CAS work. In Anna’s case line 6 actually provides the conclusion on her investigation, whereas line 4 is her problem statement. In other (empirical) cases we see both teachers and students endorsing use of CAS-related notation in a mix with algebraic notation. From a functional perspective (O’Halloran, 2005), this means that the mixed notation potentially serves purposes related to identity work and idea development work as well as functions related to pointing to the state of things in the world. The case of Anna shows that CAS do more than just the latter, i.e. point to state of affairs; CAS create identities, and do potentially influence learning.

Discussing the potential influence of mixed notation

Our previous work (Iversen, 2014), as well as the literature on mathematics education (e.g. Artigue, 2002), show that CAS can play an active and constructive role in students’ identity formation. The illustrative example presented in this paper confirms this, and further shows that mixed notation can emerge as a subsidiary practice, when CAS are implemented in the teaching of mathematics. Our analysis points to this practice not being a superfluous phenomenon, if we want to understand the way that CAS shape students’ conditions for leaning mathematics. Rather we see that mixed notation has a diverse and complex influence. Mixed notation can lead to misunderstandings, and to loss of skills regarding mathematical formalism (for related examples, see Jankvist & Misfeldt, 2015). But at the same time it is also an active part of students’ identity work and cognitive apparatus, in the sense that it opens up the students’ potential ways to express mathematics (Iversen, Misfeldt & Jankvist, in progress). We believe that the way teachers address and evaluate students’ work involving mixed notation needs to be the object for further investigation and dialogue, not least because it raises a number of important concerns for the teaching and learning of mathematics. In the following, we outline four potential points of awareness for further investigation.

Firstly, the difficulties that students often encounter when having to handle multiple representations in mathematics is well established in the semiotic approach to mathematics education (Duval, 2006). Hence, the introduction of a new notational system to be used for working in CAS is likely to lead to further difficulties for some students, especially if this notational system is introduced in a covert manner and as a superfluous and simple translation from “mathematics” to CAS notation and then back again. There is a risk that such an approach may lead to the kind of learning difficulties for students that Duval has described, i.e. that students (and in some cases teachers) see one of the representational forms as being *the* mathematical object, and the other representations (for instance the CAS notations) as being signs referring not to an abstract mathematical object, but merely to the privileged representation.

Secondly, the introduction of CAS notation is in some sense redundant, which may lead to both confusion and loss of meaning for the students. But more than that, it may contribute to the creation of new “stumbling blocks” for students, who are already experiencing difficulties related to mathematical symbols and formalism (e.g. see Niss & Jankvist, 2016). It seems easy to imagine situations, where students who are mixing CAS notation with mathematical notation ends up disabling themselves in performing, say, algebraic reductions either with paper-and-pencil or in a CAS environment. Furthermore, small discrepancies in the notations may lead to misunderstandings compromising the usual mathematical rigor. As an example, $3a$, where a is a number, is usually taken to mean $3 \cdot a$, and for that reason we regard it true that $3a = a3$. However, as part of the Danish “maths counsellor” program (see Jankvist & Niss, 2015), maths counsellors have found that several upper secondary students consider this as false, because they read $a3$ to mean a_3 due to the CAS-related convention of regarding this as such.

Thirdly, and as mentioned previously, notational transformation, including those involving mixed notation, are not consistently evaluated by teachers, and the acceptance and endorsement of “CAS notation” varies widely from teacher to teacher (Iversen, 2014). Of course, this is not unproblematic and it may potentially challenge the didactical contract (Brousseau, 1997) regarding the use of CAS in the classroom. An unclear didactical contract can lead to severe obstacles for the students, as described by Jankvist, Misfeldt and Marcussen (2016). In a situation of teacher change in a second year upper secondary mathematics class, it was observed that unclear contractual relations concerning the role of CAS fostered misguided winning strategies on the students’ behalf (in relation to Brousseau’s game metaphor), either by leading to students loss of confidence in their own mathematical skills or by causing metacognitive shifts, where the students’ focus was shifted away from the mathematical object to something else, e.g. a CAS-related procedure.

Fourth and finally, we should not forget that CAS and similar technologies change and in many respects increase the “mathematical muscles” of the students. This has both obvious and relatively well-described didactical potentials (e.g. Lagrange, 2005). If we want to capitalize on these potentials, it requires that students are able to report on their CAS activities, which is likely to involve some sort of reference to CAS notation in their mathematical writing. Taking seriously that CAS constitutes an important part of the mathematical environment for today’s students, mixed notation is also a healthy sign of students’ leaning. When students use notational elements from CAS in their written mathematical work, it may be because they are expressing mathematics in a language that they find meaningful. In that sense, CAS notation becomes a register of mathematical representation (Duval, 2006) that has relevance, and mixed notation may become a somewhat meaningful mathematical discourse. Mixed notation may assist students in clearly describing a working process involving CAS, and it may provide students with a language for expressing mathematical meaning. This “language” is of course slightly different from the standard notation – which can lead to a number of problems as described above – but nevertheless it is a language for mathematical meaning and as such writing with mixed notation may in some respects potentially enhance students’ learning of mathematics. Finally, mixed notation allows students a broader range of ways to present themselves as mathematical writers, e.g. when answering mathematical tasks. They may also present themselves as “CAS super users” (Iversen, Misfeldt & Jankvist, in progress), since mixed notation affects students’ identity work by providing a larger range of possible mathematical identities and possibilities for self-presentation (Iversen, 2014).

Concluding remarks

We argue that it is important to consider the influence of CAS in upper secondary school, and suggest that an investigation of the resulting mixed notation is indeed a relevant phenomenon to consider in future studies. Keeping in mind the growing proportion of upper secondary students who make use of ICT as a medium for writing in mathematics courses, it seems clear that the influence of CAS, and the use of mixed notation, is growing.

In the current situation in Denmark mixed notation exists, but norms and rules for accepting CAS notation as part of students' written work are neither systematically negotiated among teachers nor described in learning standards or official curricular materials. As discussed above, this can give rise to a number of difficulties for the students.

However, CAS notation is not a static thing and the technological development is promising to slowly close some of the gaps between CAS-related notation and standard algebraic notation, leaving mixed notation as a concept in flux. Still, since the potential impact of this notation covers both students' learning and their identity work, it appears highly relevant to follow closely the emergence and development of mixed notation.

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